Lattice-based sums

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Abstract

In this contribution, the well-known ordinal sum technique of posets is generalized by allowing for a lattice ordered index set instead of a linearly ordered index set, and we argue for the merits of this generalization. We will call such a proposed sum-type construction a *lattice-based sum*. Our new approach of lattice-based sum extends also the horizontal sum. We show that the lattice-based sum of posets is again a poset. Subsequently, we apply the results for constructing new lattices by investigating lattice-based sums when the summand posets are lattices. We show that under certain assumptions, the lattice-based sum of lattices will be a lattice.

Key words: Ordinal sum; Horizontal sum; Lattice-based sum; Lattice-ordered set; Posets.

1. Introduction

Several different types of algebraic structures form a background for many domains in mathematics and information sciences, such as many-valued logics, generalized measure and integral theory, quantum logics, quantum computing, etc. There are several techniques on how to build more complex structures from simpler ones. One well known technique of this type is the ordinal sums based on a linearly ordered index set. Ordinal sums appeared

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in Birkhoff's extension of partially ordered sets [1], and latter in semigroup theory by Clifford [6], see also [7, 14, 19]. Note that ordinal sums of disjoint posets in the sense of Birkhoff are also referred to as *linear sums of posets* [8]. It is worth mentioning that ordinal sums are a construction method but also a representation tool in the framework of associative conjunctions in many-valued logics and intersections in fuzzy set theory (triangular norms, see [15, 20, 21, 22, 25, 30]), as well as in the probabilistic framework of copulas modelling the dependence structure of random vectors [24, 26]. Ordinal sums have also been introduced in the frameworks of, e.g., aggregation operators [9, 11] and general algebraic structures such as hoops [2, 4]. An interesting method linking ordinal sums with real data can be found in [5].

Recall that, as pointed out in [28], due to the results of Clifford [6], see also [7, 14, 19], we know that an ordinal sum of semigroups (as introduced in [6]) whose carriers are (bounded) lattices is again a semigroup with a carrier equal to the ordinal sum (in the sense of Birkhoff [1]) of the summand lattices. However, conversely, (see [28, 29]) a straightforward application of Clifford's ordinal sum theorem to subsets of some fixed lattice L requires L to be an ordinal sum of its sublattices. But, and as shown in [28], there exist ordinal sum t-norms on bounded lattices which are not an ordinal sum of some of their sublattices (in other words, there exist ordinal sum t-norms on a bounded lattice L although L is not an ordinal sum of some of its sublattices), i.e., ordinal sum t-norms on bounded lattices need not be ordinal sums in the sense of Clifford.

Motivated by the last observation we may thus wonder whether other types of sums rather than the ordinal sums can be introduced. One possibility is the well known horizontal sum based on an unstructured index set (i.e., any two distinct indices are incomparable). Horizontal sums are exploited in the lattice theory [1, 8] and quantum structures modeling [27]. Both ordinal sums and horizontal sums were discussed in the framework of triangular norms on lattices, see, e.g., [28, 29]. Horizontal sum, however, is of a rather specific nature and imposes upon the structure of the resulting lattice to be very specific, i.e., any two distinct elements which are not involved in the same summand lattice are incomparable, and, therefore, it would be clearly desirable to explore further possibilities. Note that the structure of the lattice L heavily influences which and how many t-norms on L can be defined.

A natural question thus arises whether there is an "intermediate sum" lying between these two (extreme) cases, i.e., the ordinal sums based on a linear index set and the horizontal sums based on an unstructured index set. This paper is motivated by the later question. It is our intention to contribute to a positive answer by proposing a new type of techniques on how to build new algebraic structures from the fixed ones (possibly leading to new construction methods). We generalize the well-known ordinal sums technique of posets based on a linearly ordered index set, by allowing for lattice ordered index set. We will call such a proposed sum-type construction a lattice-based sum. Our new approach of lattice-based sums extends also the horizontal sums. We show that the lattice-based sum of posets is again a poset. Moreover, we show that if the summand posets are lattices (under certain assumptions), then the lattice-based sums will be a lattice.

Note that we have focused in this contribution on lattice-based sums of either posets or lattices as summand structures only. Further investigations of this approach could deal with lattice-based sums of semigroups (inspired by ideas of Clifford [6] in the context of ordinal sums of abstract semigroups) and therefore shifting the focus from orders to operations. This also allows us to study the theory of t-norms on bounded lattices from the point of view of lattice-based sums in analogy to what has already been done in the context of ordinal sums of t-norms (see, e.g., [15, 20, 21, 22, 25, 28, 29, 30]). We remark that other summand operations could also be taken into account (compare also, e.g., [10, 15, 20, 23]. These topics will be investigated in a future sequel to the present article.

In a recent paper of Saminger-Platz [28], there is an exhaustive description of Birkhoff's and Clifford's ordinal sums, as well as of horizontal sums, and therefore we omit this description and recommend interested readers to look on the mentioned paper. Note also, that Jipsen and Montagna (see, e.g., [16, 17, 18] and the related references therein) have introduced recently poset sums (of residuated lattices). However, the carrier of these posets sums is a subset of the Cartesian product of carriers of single summands, while the carrier of Birhoff's and Clifford's ordinal sums is the union of carriers of single summands.

Moreover, in contrast to our approach of allowing the overlapping of carriers of single summands (under certain conditions), non-overlapping of carriers of single summands in posets sums by Jipsen and Montagna excludes the possibility to generalize horizontal sums. Recall that our intention is to continue in the spirit of Birkhoff's and Clifford's ordinal sums to develop a general concept of sums extending both ordinal sums and horizontal sums.

This paper is organized as follows. In Section 2, we introduce lattice-based sums of posets. Subsequently, in Section 3, we apply the results for constructing new lattices by investigating lattice-based sums when the summand posets are lattices. We close this contribution by a short summary and further perspectives.

2. Lattice-based sums of posets

In the sequel, $(\Lambda, \sqsubseteq)$ denotes a *lattice-ordered set* in which each two-element subset $\{\alpha, \beta\}$ has an *infimum*, denoted $\inf\{\alpha, \beta\}$, and a *supremum*, denoted $\sup\{\alpha, \beta\}$. $(L_\alpha, \preceq_\alpha)$ denotes a *partially ordered set* (*poset*) for some $\alpha \in \Lambda$. The poset L_α need not have a top element nor a bottom element. When L_α has a top element and/or a bottom element, the top and bottom elements will be denoted by $\top_\alpha$ and $\perp_\alpha$, respectively. Lowercase Latin letters (e.g. "x", "y" and "z") will be used as variables ranging over the elements of L_α, and lowercase Greek letters (e.g. "α", "β" and "γ") will be used as variables ranging over the elements of Λ. If $\alpha, \beta \in \Lambda$ are incomparable elements, then we will write $\alpha \parallel \beta$. If $\alpha, \beta \in \Lambda$ such that $\alpha \sqsubseteq \beta$ but $\alpha \neq \beta$, then we will write $\alpha \sqsubset \beta$. The cardinality of a set A will be denoted by $|A|$.

Definition 1. *Consider a non-empty lattice-ordered index set* $(\Lambda, \sqsubseteq)$*. The* Λ -sum family *is a family of posets* $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ *that satisfies for all* $\alpha, \beta \in \Lambda$ *with* $\alpha \neq \beta$ *the sets* L_α *and* L_β *are either disjoint or satisfy one of the following two conditions:*

(i) $L_\alpha \cap L_\beta = \{x_{\alpha\beta}\}$ *with* $\alpha \sqsubset \beta$*, where* $x_{\alpha\beta}$ *is both the top element of* L_α *and the bottom element of* L_β*, and where for each* $\varepsilon \in \Lambda$ *with* $\alpha \sqsubset \varepsilon \sqsubset \beta$ *we have* $L_\varepsilon = \{x_{\alpha\beta}\}$*, also for all* $\delta, \gamma \in \Lambda$ *with* $\delta \parallel \gamma$*,* $\delta \sqsubset \beta$ *and* $\alpha \sqsubset \gamma$ *we have* $L_\delta = \{y_{\delta\gamma}\}$ *or* $L_\gamma = \{z_{\delta\gamma}\}$*, where* $y_{\delta\gamma}$ *is the top element of* $L_{\inf\{\delta,\gamma\}}$ *and* $z_{\delta\gamma}$ *is the bottom element of* $L_{\sup\{\delta,\gamma\}}$*.*

(ii) $1 \leq |L_\alpha \cap L_\beta| \leq 2$ *with* $\alpha \parallel \beta$*, and for each* $x_{\alpha\beta} \in L_\alpha \cap L_\beta$*,* $x_{\alpha\beta}$ *is the top element of both* L_α *and* L_β *and the bottom element of* $L_{\sup\{\alpha,\beta\}}$*, or* $x_{\alpha\beta}$ *is the bottom element of both* L_α *and* L_β *and the top element of* $L_{\inf\{\alpha,\beta\}}$*.*

If necessary, we refer to this kind of a Λ-sum family as the Λ*-sum family of posets* while the Λ*-sum family of lattices* for those whose all underlying posets L_α are lattices. In the latter case, $\{(L_\alpha, \wedge_\alpha, \vee_\alpha)\}_{\alpha\in\Lambda}$ denotes the Λ-sum family, where $\wedge_\alpha$ and $\vee_\alpha$ are the meet and join operations on L_α, respectively. As usual, the partial order relation $\preceq_\alpha$ on a lattice L_α is defined by $x \preceq_\alpha y$ if and only if $x \wedge_\alpha y = x$.

Lemma 1. *Let $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ be a Λ-sum family and $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$ with $x \neq y$.*

(i) If there exist $\alpha, \beta \in \Lambda$ such that $x \in L_\alpha$ and $y \in L_\beta$ with $\alpha \sqsubset \beta$, then for all $\alpha', \beta' \in \Lambda$ such that $x \in L_{\alpha'}$ and $y \in L_{\beta'}$ we have either $\alpha' \sqsubseteq \beta'$ or $\alpha' \parallel \beta'$.

(ii) If there exist $\alpha \in \Lambda$ such that $x, y \in L_\alpha$ with $x \preceq_\alpha y$, then for all $\alpha', \beta' \in \Lambda$ such that $x \in L_{\alpha'}$ and $y \in L_{\beta'}$ we have either $\alpha' \sqsubseteq \beta'$ or $\alpha' \parallel \beta'$.

Proof. (i) It is enough to show that if $x \in L_\alpha \cap L_{\alpha'}$ and $y \in L_\beta \cap L_{\beta'}$, where $\alpha \sqsubset \beta$ and $\beta' \sqsubset \alpha'$, then $x = y$. We distinguish the following cases:

Case 1: Suppose that $\beta' \sqsubset \beta$ and $\alpha \sqsubset \alpha'$. Since $\alpha \sqsubset \beta$ and $\beta' \sqsubset \alpha'$ it follows that $\sup\{\alpha, \beta'\} \sqsubseteq \alpha', \beta$ and hence $\sup\{\alpha, \beta'\} \sqsubseteq \inf\{\alpha', \beta\}$. Accordingly, for each $\delta \in \{\sup\{\alpha, \beta'\}, \inf\{\alpha', \beta\}\}$, we have one of the following possibilities: (a) $\beta' \sqsubset \delta \sqsubset \beta$ with $\delta = \alpha$ or $\delta = \alpha'$, (b) $\alpha \sqsubset \delta \sqsubset \alpha'$ with $\delta = \beta$ or $\delta = \beta'$, or (c) $\beta' \sqsubset \delta \sqsubset \beta$ with $\alpha \sqsubset \delta \sqsubset \alpha'$. In all these cases, it is straightforward (using Definition 1 (i)) to show that $L_\delta = \{x\} = \{y\}$, that is $x = y$.

Case 2: Suppose that $\beta' \sqsubset \beta$ and $\alpha \parallel \alpha'$. Since $x \in L_\alpha \cap L_{\alpha'}$ it follows that, by Definition 1 (ii), $x \in L_{\alpha'} \cap L_{\inf\{\alpha,\alpha'\}}$ or $x \in L_\alpha \cap L_{\sup\{\alpha,\alpha'\}}$. In the former case, since $\inf\{\alpha, \alpha'\} \sqsubset \alpha'$, we obtain that $x = y$ by a proof exactly similar to case (1) but with using $\inf\{\alpha, \alpha'\}$ instead of α in the proof. The latter case has a similar proof but with using $\sup\{\alpha, \alpha'\}$ instead of α' in the proof.

Case 3: Suppose that $\alpha \sqsubset \alpha'$ and $\beta' \parallel \beta$. This case, perfectly similar to case (2), has a similar proof.

Case 4: Suppose that $\beta \sqsubseteq \beta'$ or $\alpha' \sqsubseteq \alpha$. Therefore $\alpha \sqsubset \beta \sqsubseteq \beta' \sqsubset \alpha'$ or $\beta' \sqsubset \alpha' \sqsubseteq \alpha \sqsubset \beta$. Then we can conclude (by Definition 1 (i)) that $L_\beta = L_{\beta'} = \{x\}$ or $L_\alpha = L_{\alpha'} = \{y\}$, respectively. Then, in both cases, we get $x = y$ (since $x \in L_\alpha \cap L_{\alpha'}$ and $y \in L_\beta \cap L_{\beta'}$).

Case 5: Suppose that $\beta \parallel \beta'$ and $\alpha \parallel \alpha'$. Since $x \in L_\alpha \cap L_{\alpha'}$ it follows that, by Definition 1 (ii), $x \in L_{\alpha'} \cap L_{\inf\{\alpha,\alpha'\}}$ or $x \in L_\alpha \cap L_{\sup\{\alpha,\alpha'\}}$. Similarly, since $y \in L_\beta \cap L_{\beta'}$, it follows that $y \in L_\beta \cap L_{\inf\{\beta,\beta'\}}$ or $y \in L_{\beta'} \cap L_{\sup\{\beta,\beta'\}}$. Hence, we have one of the following four subcases:

Subcase 5(a): Suppose that $x \in L_{\alpha'} \cap L_{\inf\{\alpha,\alpha'\}}$ and $y \in L_{\beta'} \cap L_{\sup\{\beta,\beta'\}}$. By noting that $\inf\{\alpha, \alpha'\} \sqsubset \sup\{\beta, \beta'\}$, we obtain that $x = y$ by a proof exactly similar to case (1) but with using $\inf\{\alpha, \alpha'\}$ and $\sup\{\beta, \beta'\}$ instead of α and β, respectively, in the proof.

Subcase 5(b): Suppose that $x \in L_\alpha \cap L_{\sup\{\alpha,\alpha'\}}$ and $y \in L_\beta \cap L_{\inf\{\beta,\beta'\}}$. By noting that $\inf\{\beta, \beta'\} \sqsubset \sup\{\alpha, \alpha'\}$, we obtain that $x = y$ by a proof exactly similar to case (1) but with using $\sup\{\alpha, \alpha'\}$ and $\inf\{\beta, \beta'\}$ instead of α' and β', respectively, in the proof.

Subcase 5(c): Suppose that $x \in L_{\alpha'} \cap L_{\inf\{\alpha,\alpha'\}}$ and $y \in L_\beta \cap L_{\inf\{\beta,\beta'\}}$. By noting that $\inf\{\beta, \beta'\} \sqsubset \alpha'$ and $\inf\{\alpha, \alpha'\} \sqsubset \beta$, we obtain that $x = y$ by a proof exactly similar to case (1) but with using $\inf\{\alpha, \alpha'\}$ and $\inf\{\beta, \beta'\}$ instead of α and β', respectively, in the proof.

Subcase 5(d): Suppose that $x \in L_\alpha \cap L_{\sup\{\alpha,\alpha'\}}$ and $y \in L_{\beta'} \cap L_{\sup\{\beta,\beta'\}}$. By noting that $\alpha \sqsubset \sup\{\beta, \beta'\}$ and $\beta' \sqsubset \sup\{\alpha, \alpha'\}$, we obtain that $x = y$ by a proof exactly similar to case (1) but with using $\sup\{\alpha, \alpha'\}$ and $\sup\{\beta, \beta'\}$ instead of α' and β, respectively, in the proof.

(ii) First note that, by Definition 1, in case that there exists $\alpha' \neq \alpha$ such that $x \in L_\alpha \cap L_{\alpha'}$ with $x \preceq_\alpha y$ and $x \neq y$ it follows necessarily that $x = \bot_\alpha \neq y$ which implies also that either $\alpha' \sqsubseteq \alpha$ or $\alpha' \parallel \alpha$ (note that $\alpha \sqsubset \alpha'$ contradicting $x = \bot_\alpha \neq y$). Similarly, in case that there exists $\beta' \neq \alpha$ such that $y \in L_\alpha \cap L_{\beta'}$ with $x \preceq_\alpha y$ and $x \neq y$ it follows necessarily that $y = \top_\alpha \neq x$ which implies that either $\alpha \sqsubseteq \beta'$ or $\alpha \parallel \beta'$. Hence, the lemma holds trivially if $\alpha = \beta'$ or $\alpha' = \alpha$. Therefore, we assume that $\alpha' \neq \alpha$ and $\beta' \neq \alpha$. Then, in case that $\alpha \parallel \alpha'$ and $\alpha \parallel \beta'$, $x = \bot_\alpha = \bot_{\alpha'} = \top_{\inf\{\alpha,\alpha'\}}$ and $y = \top_\alpha = \top_{\beta'} = \bot_{\sup\{\alpha,\beta'\}}$ and hence, $x \in L_{\inf\{\alpha,\alpha'\}}$ and $y \in L_{\sup\{\alpha,\beta'\}}$. Since $\inf\{\alpha, \alpha'\} \sqsubset \sup\{\alpha, \beta'\}$ with $x \in L_{\alpha'}$ and $y \in L_{\beta'}$, the result follows immediately by item (i). In all other cases, i.e., $\alpha' \sqsubset \alpha$ or $\alpha \sqsubset \beta'$, the result follows immediately by item (i).

Definition 2. *Let* $(\Lambda, \sqsubseteq)$ *be a non-empty lattice-ordered index set and let* $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ *be a* Λ*-sum family. The* lattice-based sum $\bigoplus_{\alpha\in\Lambda} (L_\alpha, \preceq_\alpha)$ *is the set* $L = \bigcup_{\alpha\in\Lambda} L_\alpha$ *equipped with the order relation* $\preceq$ *defined by*

$$x \preceq y \quad \text{if and only if} \quad \begin{cases} \exists \alpha \in \Lambda \text{ such that } x, y \in L_\alpha \text{ and } x \preceq_\alpha y \\ \text{or} \\ \exists \alpha, \beta \in \Lambda \text{ such that } (x, y) \in L_\alpha \times L_\beta \text{ and } \alpha \sqsubset \beta \end{cases} \tag{1}$$

If necessary, we refer to this type of lattice-based sum as *lattice-based sums of posets.*

Remark 1. *Obviously, any (bounded) poset* $(L, \preceq)$ *can be seen as a lattice-based sum with respect to an arbitrary lattice-ordered index set* $(\Lambda, \sqsubseteq)$ *with* *top element $\top_\Lambda$, based on a trivial Λ-sum family consisting of $(L_{\top_\Lambda}, \preceq_{\top_\Lambda}) := (L, \preceq)$ and for all $\alpha \in \Lambda - \{\top_\Lambda\}$, $(L_\alpha, \preceq_\alpha) := (\{\perp_L\}, \{(\perp_L, \perp_L)\})$. To avoid such trivial situations, from now on, we will consider only lattice-based sums such that they differ from any of its summands.*

Theorem 2. *With all the assumptions of the previous definition the lattice-based sum $(L, \preceq) = \bigoplus_{\alpha\in\Lambda} (L_\alpha, \preceq_\alpha)$ is a partially ordered set.*

Proof. Obviously, the binary relation $\preceq$ is reflexive. It is also easy (by Lemma 1 and Definition 1) to prove that $\preceq$ is anti-symmetric relation. It only remains to show that $\preceq$ is transitive. Therefore, we assume that $x \preceq y$ and $y \preceq z$.

Transitivity holds trivially if either all arguments are from the same poset (i.e., $x, y, z \in L_\alpha$ for some $\alpha \in \Lambda$) or when no involved argument is an intersection point of two different posets due to the transitivity of $\preceq_\alpha$ and $\sqsubseteq$. For the remaining possibilities we distinguish the following cases:

Case 1: Suppose that there exist $\alpha, \beta \in \Lambda$ such that $x \in L_\alpha$, $y \in L_\alpha \cap L_\beta$, $z \in L_\beta$ with $x \preceq_\alpha y$ and $y \preceq_\beta z$. If $\alpha \sqsubseteq \beta$, then it is straightforward to see that $x \preceq z$. Otherwise, we have either $\beta \sqsubset \alpha$ or $\alpha||\beta$. In case that $\beta \sqsubset \alpha$, then (by Definition 1 (i)) $y = \top_\beta = \perp_\alpha$. Then we can conclude that $x = \perp_\alpha = y = \top_\beta = z$ (since $x \preceq_\alpha y$ and $y \preceq_\beta z$).

In case that $\alpha||\beta$, then (by Definition 1 (ii)) we have that $y = \top_\alpha = \top_\beta = \perp_{\sup\{\alpha,\beta\}}$ and hence $z = y = \top_\beta = \perp_{\sup\{\alpha,\beta\}}$, or we have that $y = \perp_\alpha = \perp_\beta = \top_{\inf\{\alpha,\beta\}}$ and hence $x = y = \perp_\alpha = \top_{\inf\{\alpha,\beta\}}$. Then, in both cases, we can conclude that $x \preceq z$.

Case 2: Suppose that there exist $\alpha, \beta, \delta \in \Lambda$ such that $x \in L_\alpha$, $y \in L_\alpha \cap L_\beta$, $z \in L_\delta$ with $x \preceq_\alpha y$ and $\beta \sqsubset \delta$. If $\alpha \sqsubseteq \beta$, then it is also straightforward to see that $x \preceq z$. Otherwise, we have either $\beta \sqsubset \alpha$ or $\alpha||\beta$.

In case that $\beta \sqsubset \alpha$, (by Definition 1 (i)) $y = \top_\beta = \perp_\alpha$ and hence $x = \perp_\alpha = y$ (since $x \preceq_\alpha y$). Then we can conclude that $x \preceq z$.

In case that $\alpha||\beta$, (by Definition 1 (ii)) we have two possibilities $y = \top_\alpha = \top_\beta = \perp_{\sup\{\alpha,\beta\}}$ or $y = \perp_\alpha = \perp_\beta = \top_{\inf\{\alpha,\beta\}}$. In the latter case, we can conclude that $x = \perp_\alpha = y$ and hence $x \preceq z$. In case that $y = \top_\alpha = \top_\beta = \perp_{\sup\{\alpha,\beta\}}$, we compare α and δ. Transitivity holds trivially if $\alpha \sqsubset \delta$. Otherwise, the only remaining possibility is $\alpha||\delta$. Hence, by Definition 1 (i) and noting that $y \in L_{\sup\{\alpha,\beta\}} \cap L_\beta$, $\beta \sqsubset \delta$ and $\alpha \sqsubset \sup\{\alpha, \beta\}$, we have that $L_\delta = \{\perp_{\sup\{\alpha,\delta\}}\}$ or $L_\alpha = \{\top_{\inf\{\alpha,\delta\}}\}$. Then, in both cases, it is straightforward to show that $x \preceq z$.

Case 3: Suppose that there exist $\alpha, \beta, \delta \in \Lambda$ such that $x \in L_\alpha$, $y \in L_\beta \cap L_\delta$, $z \in L_\delta$ with $y \preceq_\delta z$ and $\alpha \sqsubset \beta$. This case, perfectly similar to case (2), has a similar proof.

Case 4: Suppose that there exist $\alpha, \beta, \beta', \delta \in \Lambda$ such that $x \in L_\alpha$, $y \in L_\beta \cap L_{\beta'}$, $z \in L_\delta$ with $\alpha \sqsubset \beta$ and $\beta' \sqsubset \delta$. If $\beta \sqsubseteq \beta'$, then it is obvious that $x \preceq z$. Otherwise, we have either $\beta' \sqsubset \beta$ or $\beta' || \beta$.

In case that $\beta' \sqsubset \beta$, then we compare α and δ. Transitivity holds trivially if $\alpha \sqsubset \delta$. If $\delta \sqsubseteq \alpha$, then (by Definition 1 (i) and noting that $\beta' \sqsubset \delta \sqsubseteq \alpha \sqsubset \beta$, $y \in L_\beta \cap L_{\beta'}$) $L_\delta = L_\alpha = \{y\}$ and hence $x = y = z$. Otherwise, the only remaining possibility is $\alpha || \delta$. Hence, by Definition 1 (i) and noting that $\beta' \sqsubset \delta$, $\alpha \sqsubset \beta$, $y \in L_\beta \cap L_{\beta'}$, we have that $L_\delta = \{\perp_{\sup\{\alpha,\delta\}}\}$ or $L_\alpha = \{\top_{\inf\{\alpha,\delta\}}\}$. Then, in both cases, it is straightforward to show that $x \preceq z$.

In case that $\beta' || \beta$, then (by Definition 1 (ii)) we have two possibilities $y = \top_{\beta'} = \top_\beta = \perp_{\sup\{\beta',\beta\}}$ or $y = \perp_{\beta'} = \perp_\beta = \top_{\inf\{\beta',\beta\}}$. Hence, $y \in L_{\sup\{\beta,\beta'\}} \cap L_{\beta'}$ or $y \in L_{\inf\{\beta,\beta'\}} \cap L_\beta$. So, we compare α and δ. Transitivity holds trivially if $\alpha \sqsubset \delta$. Otherwise, the only remaining possibility is $\alpha || \delta$. Hence, by Definition 1 (i) and noting that $\alpha \sqsubset \beta \sqsubset \sup\{\beta', \beta\}$, $\beta' \sqsubset \delta$, $\alpha \sqsubset \beta$, and $\inf\{\beta', \beta\} \sqsubset \beta' \sqsubset \delta$, we have that $L_\delta = \{\perp_{\sup\{\alpha,\delta\}}\}$ or $L_\alpha = \{\top_{\inf\{\alpha,\delta\}}\}$. Then, in both cases, it is straightforward to show that $x \preceq z$.

We can easily check that, if the index set is linear, then the lattice-based sum reduces to the ordinal sum, i.e., we obtain the ordinal sum of posets in the sense of Birkhoff in which any two posets overlap in at most one point. On the other hand, Proposition 3 clarifies the relationship between our lattice-based sums and the well-known *horizontal sum* technique.

Recall that a bounded poset $(L, \preceq, 0, 1)$ is called a *horizontal sum* of the bounded posets $((L_i, \preceq_i, 0, 1))_{i \in I}$ if $L = \bigcup_{i \in I} L_i$ with $L_i \cap L_j = \{0, 1\}$ whenever $i \neq j$, and $x \preceq y$ if and only if there is an $i \in I$ such that $\{x, y\} \subseteq L_i$ and $x \preceq_i y$ (compare, e.g., horizontal sums of effect algebras [27]).

The proof of the next proposition is obvious:

Proposition 3. *Let $(L, \preceq, 0, 1)$ be a bounded poset. Then the following are equivalent:*

- *(i)* $(L, \preceq, 0, 1)$ *is a horizontal sum of the bounded posets* $((L_i, \preceq_i, 0, 1))_{i \in I}$.
- *(ii)* $(L, \preceq, 0, 1)$ *is a lattice-based sum of the bounded posets* $((L_\alpha, \preceq_\alpha, 0, 1))_{\alpha \in \Lambda}$, *where* $(\Lambda, \sqsubseteq)$ *is the lattice in which* Λ *is the set* I *with two more elements* $\perp$ *and* $\top$ *such that* $L_\perp = \{0\}$ *and* $L_\top = \{1\}$, *and the partial order* $\sqsubseteq$ *is defined on* Λ *by: for all* $\alpha \in \Lambda$, $\perp \sqsubseteq \alpha$ *and* $\alpha \sqsubseteq \top$.

Accordingly, both the ordinal sums of posets in the sense of Birkhoff and horizontal sums of the bounded posets are particular cases of our more general approach of lattice-based sums of posets.

Note that the strategy described in this section focuses on the union of the carriers and an order consistent (i.e., in agreement) with both the order of the underlying posets and the order of the lattice-ordered index set (see Definition 2). Accordingly, the order relation for elements from different summand carriers is inherited from the lattice-ordered index set. We end this section by showing some examples to clarify our ideas.

Example 1. *Consider the lattice-ordered index* $(\Lambda, \sqsubseteq)$ *in Figure 1. It is easy to check that each of the families associated with the structures in Figures 2, 3, 4 and 5 forms a* Λ*-sum family and hence each of these structures is a lattice-based sum of posets. Note that, in Figure 4,* L_δ *is the singleton poset* $\{x\}$*, where* x *is the bottom element of* L_β *and the top element* $L_{\perp_\Lambda}$*, since* $L_\beta \cap L_{\perp_\Lambda} = \{x\}$ *and* $\perp_\Lambda \sqsubset \delta \sqsubset \beta$*.*

Example 2. *Consider the lattice-ordered index set in Figure 6. The family of non-trivial posets (i.e., they are not singletons, compare also Remark 1) associated with the structure in Figure 7 is not a* Λ*-sum family and hence the structure is not a lattice-based sum because* $L_\alpha \cap L_\beta = \{x_{\alpha\beta}\}$*, with* $x_{\alpha\beta} = \top_\alpha = \perp_\beta$*,* $\delta \sqsubset \beta$*,* $\alpha \sqsubset \gamma$*, but neither* $L_\delta = \{\top_{\inf\{\delta,\gamma\}}\}$ *nor* $L_\gamma = \{\perp_{\sup\{\delta,\gamma\}}\}$*. Note that, although the structure in Figure 7 is a poset, its order relation is not consistent with the order of the index set (since, for* $x \in L_\delta$ *and* $y \in L_\gamma$*,* $x \preceq y$ *while the only elements* δ *and* γ *in the index set associated with* x *and* y*, respectively, are incomparable elements in* Λ*). A slight modification of the family associated with the structure in Figure 7 by putting* $L_\delta = \{\top_{\inf\{\delta,\gamma\}}\}$ *produces the* Λ*-sum family of posets associated with the structure in Figure 8 which is a lattice-based sum. Note that in this case the consistency holds, namely for* x *and* y *as above, we have* $x \preceq y$*,* $x \in L_\delta \cap L_{\perp_\Lambda}$ *and* $y \in L_\gamma$*, and hence there exists* $\perp_\Lambda, \gamma \in \Lambda$ *associated with* x *and* y*, respectively, such that* $\perp_\Lambda \sqsubset \gamma$*.*

Example 3. *Consider the lattice-ordered index set* $(\Lambda, \sqsubseteq)$ *in Figure 9. The family of posets associated with the structure in Figure 10 is not* Λ*-sum family since it violates the condition (ii) in Definition 1 where* $x_{\alpha\delta} \in L_\alpha \cap L_\delta$ *is the bottom element of both* L_α *and* L_δ *while* $x_{\alpha\delta}$ *is not the top element of* $L_{\inf\{\alpha,\delta\}}$*. A slight modification of the family associated with the structure in Figure 10*

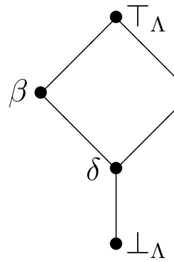

Figure 1: Lattice $(\Lambda, \sqsubseteq)$

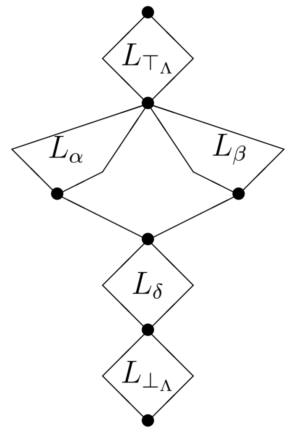

Figure 2: Family 1

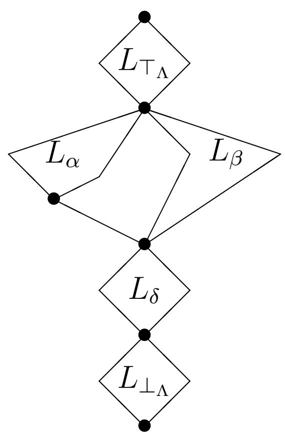

Figure 3: Family 2

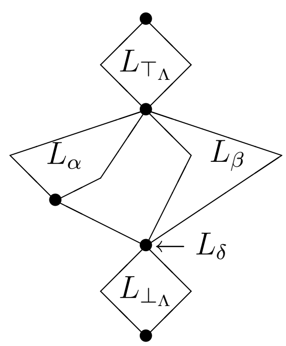

Figure 4: Family 3

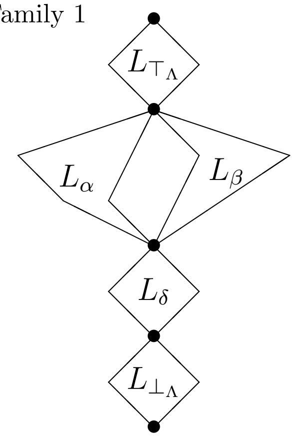

Figure 5: Family 4

by putting $L_\gamma = L_\epsilon = \{x_{\alpha\delta}\}$, where $x_{\alpha\delta}$ is the bottom element of both L_α and L_δ, produces the Λ-sum family of posets associated with the structure in Figure 11 which is a lattice-based sum.

Example 4. *Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ in Figure 12. It is routine to check that the two families of posets associated with the structures in Figure 13 and Figure 14 are Λ-sum families since they satisfy all the conditions in Definition 1. For example, we find that $L_{\alpha_1} = L_{\beta_1} = L_{\delta_1} = \{x\}$ where x is the top element of $L_{\bot_\Lambda}$. This is because L_{β_2} overlaps with L_{δ_2} where $\alpha_1 \sqsubset \delta_1 \sqsubset \delta_2$ and $\beta_2 \sqsubset \beta_3$, moreover, L_{α_2} overlaps with L_{δ_3} where $\beta_1 \sqsubset \delta_1 \sqsubset \delta_3$ and $\alpha_2 \sqsubset \alpha_3$.*

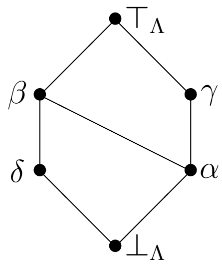

Figure 6: Lattice $(\Lambda, \sqsubseteq)$

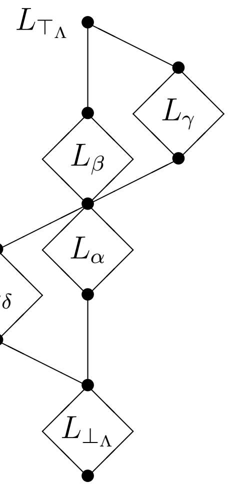

Figure 7: Not Family

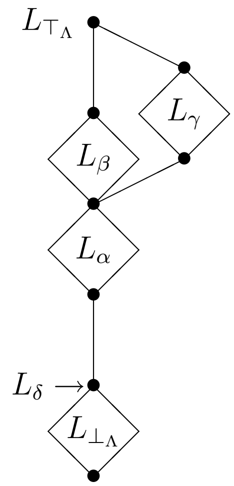

Figure 8: Family

3. Lattice-based sums of lattices

Definition 3. *Given a Λ-sum family $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ and $x \in \bigcup_{\alpha\in\Lambda} L_\alpha$. We say that an element $\alpha^* \in \Lambda$ is* a maximal index of x *(respectively, $\alpha_* \in \Lambda$ is a* minimal index of x*) if α^* is a maximal (respectively, minimal) element of the set $I_x = \{\alpha \in \Lambda \mid x \in L_\alpha\}$. Denote by $I_x^{\max}$ and $I_x^{\min}$ the set of all maximal and minimal indices of x, respectively.*

Obviously, if $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ is a Λ-sum family with finite lattice index set Λ , then, for all $x \in \bigcup_{\alpha\in\Lambda} L_\alpha$, the set $I_x = \{\alpha \in \Lambda : x \in L_\alpha\}$ contains maximal and minimal elements. For example, in the Λ-sum family in Figure 14, if x is the top element of L_{β_2} then $I_x = \{\beta_2, \beta_3, \beta_4, \delta_2\}$, $I_x^{\max} = \{\beta_4\}$ and $I_x^{\min} = \{\beta_2\}$. Note that, in general, the set I_x need not have maximal or minimal elements. For example,

Example 5. *Consider the infinite lattice index set*

$$\Lambda = \{1 - 1/n \mid n \text{ is a natural number}\} \cup \{1\}$$

and the family of posets $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ with $L_0 = \{x\}$, $L_{1-1/n} = \{y\}$, for all $n > 1$, and $L_1 = \{z\}$, where $x \neq y \neq z$. Hence, by Definition 1, the later family is a Λ-sum family. However, the set I_y=$\{1 - 1/n \mid n > 1\}$ does not have a maximal element.

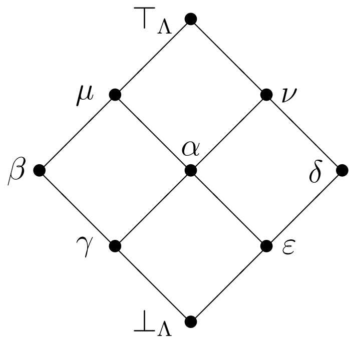

Figure 9: Lattice $(\Lambda, \sqsubseteq)$

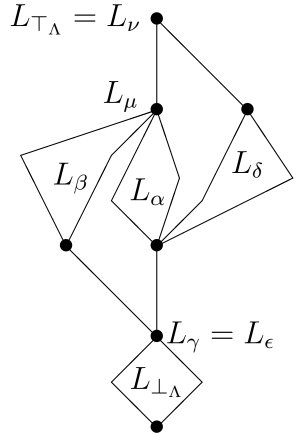

Figure 10: not Family

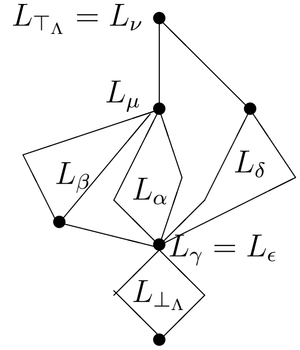

Figure 11: Family

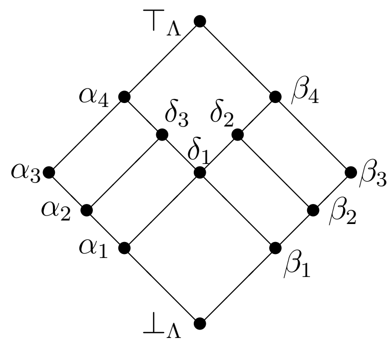

Figure 12: Lattice $(\Lambda, \sqsubseteq)$

Given a Λ-sum family $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$. Let $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$ with $x \neq y$. If for all $\alpha, \beta \in \Lambda$ such that $x \in L_\alpha$ and $y \in L_\beta$ we have $\alpha \parallel \beta$, then we will write $x \parallel y$. If $x, y \in L_\alpha$ for some $\alpha \in \Lambda$ such that $x \not\preceq_\alpha y$ and $y \not\preceq_\alpha x$, then we will write $x \parallel_\alpha y$. Obviously, x and y are incomparable if $x \parallel y$ or $x \parallel_\alpha y$ for some $\alpha \in \Lambda$.

Definition 4. *A Λ-sum family $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ is said to be* with (M) condition *if it satisfies the following* (M) *condition:*

(M)*: for all $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$ with $x \parallel y$, the sets $I_x = \{\delta \in \Lambda : x \in L_\delta\}$ and $I_y = \{\beta \in \Lambda : y \in L_\beta\}$ have both maximal and minimal elements.*

Example 6. *The following are examples of Λ-sum families with* (M) *condi-*

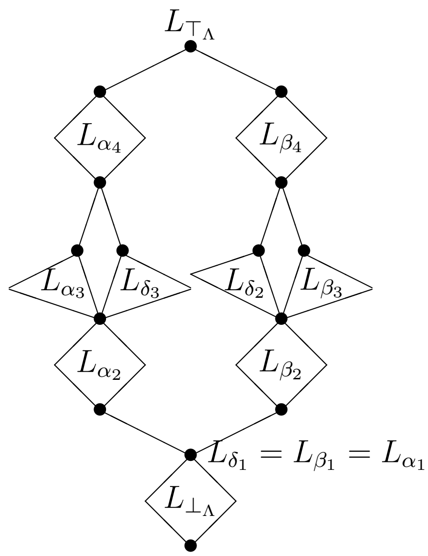

Figure 13: Family 1

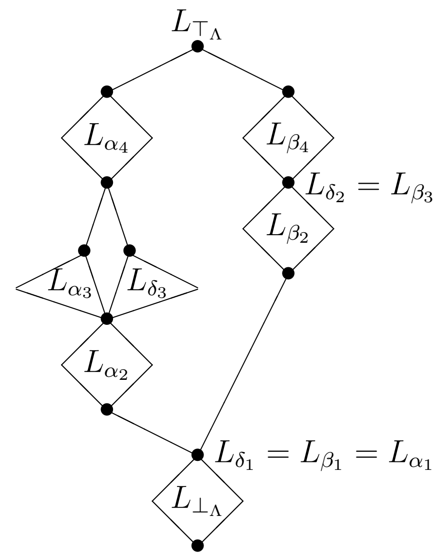

Figure 14: Family 2

tion:

(i) A Λ-sum family with finite index set Λ is a Λ-sum family with (M) *condition.*

(ii) A Λ-sum family satisfying, for all $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$ with $x \parallel y$, that both I_x and I_y are finite is a Λ-sum family with (M) *condition, e.g., a family $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ of pairwise disjoint posets.*

(iii) A Λ-sum family satisfying, for all $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$ with $x \parallel y$, that every chain in I_x and I_y has an upper and lower bound in I_x and I_y, respectively, is a Λ-sum family with (M) *condition.*

Definition 5. *A* semibounded Λ-sum family of posets (lattices) *is a Λ-sum family of posets (lattices) that satisfies, for all $\alpha, \beta \in \Lambda$ with $\alpha \parallel \beta$, the set $L_{\inf\{\alpha,\beta\}}$ has a top element and the set $L_{\sup\{\alpha,\beta\}}$ has a bottom element.*

We know that the set I_x need not have maximal nor minimal elements. Nevertheless, given an element x, we shall write that α' is a maximal (minimal) index of x meaning that the set I_x has a maximal (minimal) element which is equal to α'.

Lemma 4. *Let $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ be a semibounded Λ-sum family of posets. For all $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$ with $x \parallel y$, it holds that:*

(i) If $x \in L_\delta$ for some $\delta \in \Lambda$ and α^*, β^* are maximal indices of y, then $\top_{\inf\{\delta,\alpha^*\}} = \top_{\inf\{\delta,\beta^*\}}$.

(ii) If $x \in L_\delta$ for some $\delta \in \Lambda$ and α_*, β_* are minimal indices of y, then $\bot_{\sup\{\delta,\alpha_*\}} = \bot_{\sup\{\delta,\beta_*\}}$.

Proof. We shall prove only the item (i). The second item (ii), perfectly dual to item (i), has a dual proof.

(i) It is obvious that if $\alpha^* = \beta^*$ then the lemma holds. So, assume that $\alpha^* \neq \beta^*$. Since $x \parallel y$ then $\delta \parallel \alpha^*$ and $\delta \parallel \beta^*$. Moreover, by maximality of α^* and β^*, we can conclude that $\alpha^* \parallel \beta^*$. Therefore, by Definition 1 (ii), $y = \bot_{\alpha^*} = \bot_{\beta^*} = \top_{\inf\{\alpha^*,\beta^*\}}$ and hence $y \in L_{\inf\{\alpha^*,\beta^*\}} \cap L_{\alpha^*}$.

We distinguish the following two cases:

Case 1: Suppose that $(\inf\{\delta,\alpha^*\}) \parallel \beta^*$, i.e. $\inf\{\delta,\alpha^*\}$ and β^* are incomparable.

Since $y \in L_{\inf\{\alpha^*,\beta^*\}} \cap L_{\alpha^*}$, $\inf\{\delta,\alpha^*\} \sqsubset \alpha^*$ and $\inf\{\alpha^*,\beta^*\} \sqsubset \beta^*$, hence, by Definition 1 (i), $L_{\inf\{\delta,\alpha^*\}} = \left\{\top_{\inf\{\alpha^*,\beta^*,\delta\}}\right\}$ (note that the other possibility which is $L_{\beta^*} = \left\{\bot_{\sup\{\inf\{\alpha^*,\delta\},\beta^*\}}\right\}$ contradicting the maximality of β^*). We have one of the following two subcases:

Subcase $1(a)$: Suppose that $(\inf\{\delta,\beta^*\}) \parallel \alpha^*$. Similar to Case 1, $L_{\inf\{\delta,\beta^*\}} = \left\{\top_{\inf\{\alpha^*,\beta^*,\delta\}}\right\}$. Thus, $L_{\inf\{\delta,\alpha^*\}} = L_{\inf\{\delta,\beta^*\}}$ and hence $\top_{\inf\{\delta,\alpha^*\}} = \top_{\inf\{\delta,\beta^*\}}$.

Subcase $1(b)$: Suppose that $\inf\{\delta,\beta^*\}$and α^* are comparable. If $\alpha^* \sqsubseteq \inf\{\delta,\beta^*\}$, also $\alpha^* \sqsubseteq \inf\{\delta,\beta^*\} \sqsubset \beta^*$ contradicting the incomparability to β^*. Therefore, let $\inf\{\delta,\beta^*\} \sqsubset \alpha^*$, from which we can conclude that $\inf\{\alpha^*,\beta^*,\delta\} = \inf\{\delta,\beta^*\} \sqsubseteq \inf\{\delta,\alpha^*\}$. By this and since $L_{\inf\{\delta,\alpha^*\}} = \left\{\top_{\inf\{\alpha^*,\beta^*,\delta\}}\right\}$, we can conclude that $\top_{\inf\{\delta,\alpha^*\}} = \top_{\inf\{\delta,\beta^*\}}$.

Case 2: Suppose that $\inf\{\delta,\alpha^*\}$ and β^* are comparable. Hence, $\inf\{\delta,\alpha^*\} \sqsubset \beta^*$ (otherwise, i.e. $\beta^* \sqsubseteq \inf\{\delta,\alpha^*\}$, implies that $\beta^* \sqsubset \alpha^*$, contradiction (since $\alpha^* \parallel \beta^*$)). Thus, $\inf\{\alpha^*,\beta^*,\delta\} = \inf\{\delta,\alpha^*\} \sqsubseteq \inf\{\delta,\beta^*\}$. We have one of the following two subcases:

Subcase $2(a)$: Suppose that $\inf\{\delta,\beta^*\}$ and α^* are comparable. This implies that $\inf\{\delta,\beta^*\} \sqsubseteq \inf\{\delta,\alpha^*\}$ (as it is proved in the Subcase 1(b)). Thus, we obtain $\inf\{\delta,\beta^*\} = \inf\{\delta,\alpha^*\}$ and hence, $\top_{\inf\{\delta,\alpha^*\}} = \top_{\inf\{\delta,\beta^*\}}$.

Subcase $2(b)$: Suppose that $(\inf\{\delta,\beta^*\}) \parallel \alpha^*$. This implies that $L_{\inf\{\delta,\beta^*\}} = \left\{\top_{\inf\{\alpha^*,\beta^*,\delta\}}\right\}$ (as it is proved in the Subcase 1(a)). Since $\inf\{\alpha^*,\beta^*,\delta\} = \inf\{\delta,\alpha^*\} \sqsubseteq \inf\{\delta,\beta^*\}$, then we can conclude that $\top_{\inf\{\delta,\alpha^*\}} = \top_{\inf\{\delta,\beta^*\}}$.

Corollary 1. *Let $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ be a semibounded Λ-sum family of posets. Let $x, y \in \bigcup_{\alpha\in\Lambda} L_\alpha$ with $x \parallel y$, and assume that the sets I_x and I_y have both maximal and minimal elements. Then*

(i) For all $\delta_1, \delta_2 \in I_x^{\max}$ and $\beta_1, \beta_2 \in I_y^{\max}$, $\top_{\inf\{\delta_1,\beta_1\}} = \top_{\inf\{\delta_2,\beta_2\}}$.
(ii) For all $\delta_1, \delta_2 \in I_x^{\min}$ and $\beta_1, \beta_2 \in I_y^{\min}$, $\perp_{\sup\{\delta_1,\beta_1\}} = \perp_{\sup\{\delta_2,\beta_2\}}$.

Proof. (i) By Lemma 4 (i), $\top_{\inf\{\delta_1,\beta_1\}} = \top_{\inf\{\delta_1,\beta_2\}}$ and $\top_{\inf\{\delta_1,\beta_2\}} = \top_{\inf\{\delta_2,\beta_2\}}$. Hence, $\top_{\inf\{\delta_1,\beta_1\}} = \top_{\inf\{\delta_2,\beta_2\}}$. The second item (ii), perfectly dual to item (i), has a dual proof (using Lemma 4 (ii)).

Lemma 5. *Let $(L, \preceq)$ be a lattice-based sum of a Λ -sum family $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ and let $x, z \in L$ with $z \preceq x$. Then*

(i) For any maximal index α^* of x there exists $\delta \in \Lambda$ such that $z \in L_\delta$ and $\delta \sqsubseteq \alpha^*$.
(ii) For any minimal index δ_* of z there exists $\alpha \in \Lambda$ such that $x \in L_\alpha$ and $\delta_* \sqsubseteq \alpha$.

Proof. We shall prove only the item (i). The second item (ii), perfectly dual to item (i), has a dual proof. Suppose that α^* is a maximal index of x and $z \preceq x$. Then there exists $\alpha, \delta \in \Lambda$ such that $x \in L_\alpha$, $z \in L_\delta$ and ($\delta \sqsubset \alpha$ or $\delta = \alpha$ with $z \preceq_\alpha x$). The lemma trivially holds if $\alpha \sqsubseteq \alpha^*$. Note that $\alpha^* \sqsubset \alpha$ contradicts the maximality of α^*. Also, $\alpha \parallel \alpha^*$ leads to either $\delta \sqsubset \alpha^*$ (hence the lemma trivially holds) or $\delta \parallel \alpha^*$. Therefore we demand that $\alpha \parallel \alpha^*$ and $\delta \parallel \alpha^*$. Thus, by Definition 1 (ii) and since α^* is a maximal index of x, $x = \perp_\alpha = \perp_{\alpha^*} = \top_{\inf\{\alpha,\alpha^*\}}$ and hence $x \in L_\alpha \cap L_{\inf\{\alpha,\alpha^*\}}$. In case that $\delta = \alpha$ and $z \preceq_\alpha x$, we have $z = \perp_\alpha = \top_{\inf\{\alpha,\alpha^*\}} = x$ (since $z \preceq_\alpha x$). Hence, there exists $\delta' = \inf\{\alpha, \alpha^*\} \in \Lambda$ such that $z \in L_{\delta'}$ and $\delta' \sqsubset \alpha^*$.

In case that $\delta \sqsubset \alpha$, we have that $L_\delta = \{z = \top_{\inf\{\delta,\alpha^*\}}\}$ (by Definition 1 (i) and since $\inf\{\alpha, \alpha^*\} \sqsubset \alpha$, $x \in L_\alpha \cap L_{\inf\{\alpha,\alpha^*\}}$, $\delta \sqsubset \alpha$ and $\delta \parallel \alpha^*$). Note that the other possibility which is $L_{\alpha^*} = \left\{\perp_{\sup\{\alpha^*,\delta\}} = x\right\}$ contradicting the maximality of α^*. Hence, there exists $\delta' = \inf\{\delta, \alpha^*\} \in \Lambda$ such that $z \in L_{\delta'}$ and $\delta' \sqsubset \alpha^*$. This completes the proof.

The assumption of maximality and minimality in Lemma 4, Lemma 5 and Corollary 1 above are indispensable for the validity of the mentioned results as the following example shows.

Example 7. *Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ in Figure 12 and its Λ-sum family in Figure 15 (we leave for the reader to check that the family in Figure 15 is a Λ-sum family). Let x be the top of L_{β_2} and y be the top of L_{α_2}. It is obvious that $x \parallel y$ where $I_x = \{\beta_2, \delta_2\}$ and $I_y = \{\alpha_2, \alpha_3, \delta_3\}$. For $\beta_2, \delta_2 \in I_x$ and $\alpha_2, \delta_3 \in I_y$ (note that β_2 and α_2 are not maximal), we have that $\top_{\inf\{\beta_2,\delta_3\}} = \top_{\beta_1} \neq \top_{\alpha_1} = \top_{\inf\{\delta_2,\alpha_2\}}$. Of course, as already proved in Lemma 4, replacing α_2 by the maximal α_3 and replacing β_2 by the maximal δ_2 renders the equality, i.e., $\top_{\inf\{\delta_2,\delta_3\}} = \top_{\delta_1} = \top_{\inf\{\delta_2,\alpha_3\}}$. On the other hand, let z be the top of L_{α_1} and x as before, then it is obvious that $z \preceq x$. Now, for $\beta_2 \in I_x$, where β_2 is not a maximal, we see that neither $\delta_1 \sqsubseteq \beta_2$ nor $\alpha_1 \sqsubseteq \beta_2$, where $I_z = \{\alpha_1, \delta_1\}$.*

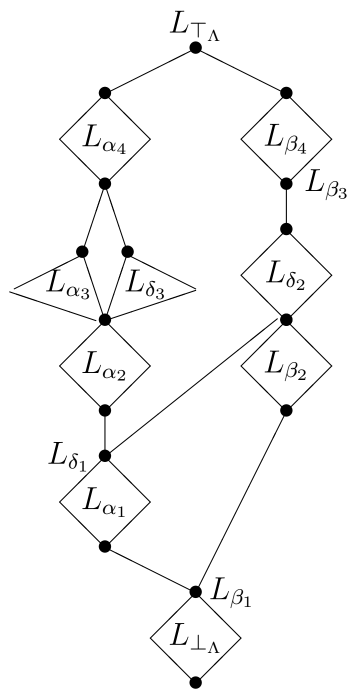

Figure 15: Family

Note that, for x and y as in the above example, $inf\{x, y\} = \top_{\inf\{\delta_2,\delta_3\}} = \top_{\delta_1} = \top_{\inf\{\delta_2,\alpha_3\}}$ where α_3 and δ_3 are maximal indices of y while δ_2 is the maximal index of x. Although, $\top_{\inf\{\beta_2,\delta_3\}} = \top_{\beta_1} = \top_{\inf\{\beta_2,\alpha_3\}}$ where β_2 is not maximal, we see that $\inf\{x, y\} \neq \top_{\inf\{\beta_2,\delta_3\}} = \top_{\inf\{\beta_2,\alpha_3\}}$. Inspired by the last observation we introduce the following definition.

Definition 6. *Let* $(\Lambda, \sqsubseteq)$ *be a non-empty lattice-ordered index set and let* $\{(L_\alpha, \wedge_\alpha, \vee_\alpha)\}_{\alpha\in\Lambda}$ *be a semibounded* Λ*-sum family of lattices with* (M) *condition. Put* $L = \bigcup_{\alpha\in\Lambda} L_\alpha$ *, for every* $x \in L$*, denote by* $I_x^{\max}$ *and* $I_x^{\min}$ *the set of all maximal and minimal indices of* x*, respectively, and define the binary operations* $\wedge$ *and* $\vee$ *on* L *by*

$$x \wedge y = \begin{cases} x \wedge_\alpha y & \text{if } (x,y) \in L_\alpha \times L_\alpha, \\ x & \text{if } (x,y) \in L_\alpha \times L_\beta \text{ and } \alpha \sqsubset \beta, \\ y & \text{if } (x,y) \in L_\alpha \times L_\beta \text{ and } \beta \sqsubset \alpha, \\ \top_{\inf\{\alpha^*,\beta^*\}} & \text{if } x \parallel y,\ \alpha^* \in I_x^{\max} \text{ and } \beta^* \in I_y^{\max}. \end{cases} \tag{2}$$

$$x \vee y = \begin{cases} x \vee_\alpha y & \text{if } (x,y) \in L_\alpha \times L_\alpha, \\ y & \text{if } (x,y) \in L_\alpha \times L_\beta \text{ and } \alpha \sqsubset \beta, \\ x & \text{if } (x,y) \in L_\alpha \times L_\beta \text{ and } \beta \sqsubset \alpha, \\ \bot_{\sup\{\alpha_*,\beta_*\}} & \text{if } x \parallel y,\ \alpha_* \in I_x^{\min} \text{ and } \beta_* \in I_y^{\min}. \end{cases} \tag{3}$$

Then we say that $(L, \wedge, \vee)$ *is the* lattice-based sums *of all* $\{(L_\alpha, \wedge_\alpha, \vee_\alpha)\}_{\alpha\in\Lambda}$*. If necessary, we refer to this type of lattice-based sum as* lattice-based sums of lattices.

Example 8. *Consider the lattice-ordered index set* $(\Lambda, \sqsubseteq)$ *in Figure 16 and consider the family of posets associated with the structure in Figure 17. Under the assumption that the summand posets are (semi)bounded lattices, it is easy to check that the family in Figure 17 is a* Λ*-sum family and satisfies the condition (M) (since the index set is finite). Moreover, it is easy to see that the structure in Figure 17 is a lattice where the meet and join are defined as in (2) and (3), respectively. For the reader convenience, we check only the case when we have two incomparable elements. Therefore, consider* $x \parallel y$ *where* $x = \top_{\alpha_4}$ *and* $y = \top_{\beta_4}$*. Obviously,* $I_x = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ *and* $I_y = \{\beta_0, \beta_1, \beta_2, \beta_3, \beta_4\}$*. Hence,* $I_x^{\max} = \{\alpha_0, \alpha_1\}$ *and* $I_y^{\max} = \{\beta_0, \beta_1\}$*. For computing* $x \wedge y$*, the last case in (2) applies. By noting that* $L_{\delta_1} = L_{\delta_2} = L_{\delta_3} = \{\top_{\delta_4}\}$*, we find that, for each* $\alpha^* \in I_x^{\max}$ *and* $\beta^* \in I_y^{\max}$*,* $x \wedge y = \top_{\inf\{\alpha^*,\beta^*\}} = \top_{\delta_4}$ *and hence it produces the same result. Note that if* $\alpha^* \in I_x$ *is not maximal or* $\beta^* \in I_y$ *is not maximal,* $x \wedge y$ *need not be equal to* $\top_{\inf\{\alpha^*,\beta^*\}}$*, e.g., for* $\alpha_4 \in I_x$ *and* $\beta_1 \in I_y$*, we have* $\top_{\inf\{\alpha_4,\beta_1\}} = \top_{\alpha_6} \neq x \wedge y$*.*

In the above example, we see that the lattice-based sum of lattices is a lattice. The following theorem shows that this is always true.

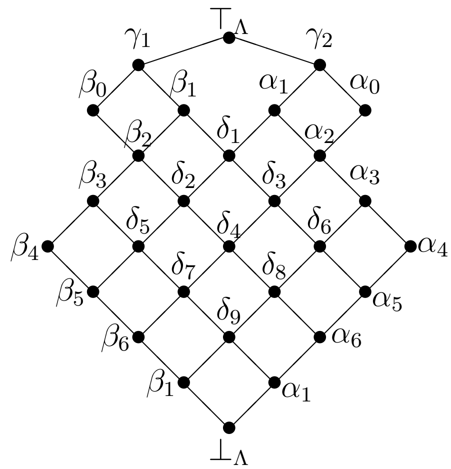

Figure 16: Lattice $(\Lambda, \sqsubseteq)$

Theorem 6. *With all the assumptions of Definition 6 the lattice-based sum* $(L, \wedge, \vee) = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \wedge_\alpha, \vee_\alpha)$ *is a lattice.*

Proof. The proof runs only for the operation $\wedge$. The operation $\vee$, perfectly dual to the operation $\wedge$, has a dual proof.

First it is necessary to check that the operation $\wedge$ is well-defined. If $x \parallel y$, then the last case in (2) apply. By Corollary 1 (i), $x \wedge y$ produces the same result for each $\alpha^* \in I_x^{\max}$ and $\beta^* \in I_y^{\max}$. Otherwise (i.e., $x \nparallel y$), by Lemma 1, a problem can only arise if $(x, y) \in L_\alpha \times L_\beta$ and, say, $\alpha \sqsubset \beta$ and $x \in L_\alpha \cap L_\beta$. In this situation the first two cases in (2) apply. But, by Definition 1 (i), then $x = \top_\alpha = \perp_\beta$, i.e., for each $y \in L_\beta$ we get $x \wedge_\beta y = x$, thus producing the same result in either case. This is true even if there exists $\gamma \in \Lambda$ with $\alpha \sqsubset \gamma \sqsubset \beta$ and $L_\gamma = \{x\}$. The situations $y \in L_\alpha \cap L_\beta$ and $\beta \sqsubset \alpha$ are checked in complete analogy.

Recall that, by Theorem 2 and noting that the partial order relation $\preceq_\alpha$ on a lattice L_α is defined by $x \preceq_\alpha y$ if and only if $x \wedge_\alpha y = x$, the lattice-based sum $(L, \preceq) = \bigoplus_{\alpha \in \Lambda} L_\alpha$ is a poset, whereby the order relation $\preceq$ is given by (1). Now, we need to prove that the operation $\wedge$ in (2) is a meet operation on L, i.e. for all $x, y \in L$, $x \wedge y$ is the infimum in $(L, \preceq)$ of the set $\{x, y\}$. It is straightforward to show that $x \wedge y$ is a lower bound in $(L, \preceq)$ for the set $\{x, y\}$. Let $z \in L$ be another lower bound for $\{x, y\}$. Then, we need only to show that, in case that $x \parallel y$, $z \preceq x \wedge y$. So suppose that $x \parallel y$, $z \preceq x$ and $z \preceq y$. Then, $x \wedge y = \top_{\inf\{\alpha^*, \beta^*\}}$, where $\alpha^* \in I_x^{\max}$ and $\beta^* \in I_y^{\max}$, and hence, by

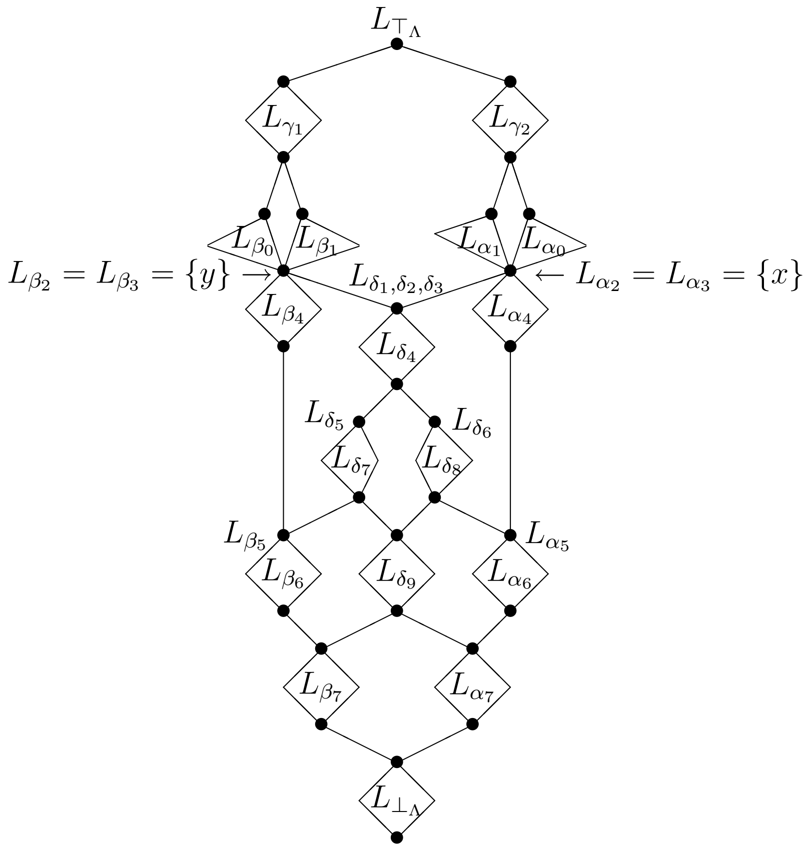

Figure 17: Family

Lemma 5 (i), there exist $\delta, \delta' \in \Lambda$ such that $z \in L_\delta \cap L_{\delta'}$, $\delta \sqsubseteq \alpha^*$ and $\delta' \sqsubseteq \beta^*$. If δ and δ' are comparable, then $\delta \sqsubseteq \delta'$, say, and hence $\delta \sqsubseteq \inf\{\alpha^*, \beta^*\}$. Then we can conclude that $z \preceq x \wedge y$. Otherwise, i.e. if $\delta \parallel \delta'$, then (by Definition 1 (ii)) $z = \bot_\delta = \bot_{\delta'} = \top_{\inf\{\delta,\delta'\}}$ or $z = \top_\delta = \top_{\delta'} = \bot_{\sup\{\delta,\delta'\}}$. Note that the latter case is not possible if $\delta = \alpha^*$ or $\delta' = \beta^*$ since it is contradicting the maximality of α^* and β^*, respectively. Therefore, in case that $z = \top_\delta = \top_{\delta'} = \bot_{\sup\{\delta,\delta'\}}$, we demand that $\delta \neq \alpha^*$ and $\delta' \neq \beta^*$. Then, in this case, we compare δ and β^*. Since $\alpha^* \parallel \beta^*$, we have either $\delta \sqsubset \beta^*$ or $\delta \parallel \beta^*$. If $\delta \sqsubset \beta^*$, then $\delta \sqsubseteq \inf\{\alpha^*, \beta^*\}$ and hence we can conclude that $z \preceq x \wedge y$. If $\delta \parallel \beta^*$, then $L_\delta = \{\top_{\inf\{\delta,\beta^*\}} = z\}$ (by Definition 1 (i) and the maximality of β^* with noting that $z \in L_{\delta'} \cap L_{\sup\{\delta,\delta'\}}$, $\delta \sqsubset \sup\{\delta, \delta'\}$ and $\delta' \sqsubset \beta^*$) and hence $z \in L_{\inf\{\delta,\beta^*\}}$. Since $\inf\{\delta, \beta^*\} \sqsubseteq \inf\{\alpha^*, \beta^*\}$, then it is straightforward to see that $z \preceq x \wedge y$. In case that $z = \bot_\delta = \bot_{\delta'} = \top_{\inf\{\delta,\delta'\}}$,

$z \in L_{\inf\{\delta,\delta'\}}$. Since $\inf\{\delta, \delta'\} \sqsubseteq \inf\{\alpha^*, \beta^*\}$, then we can conclude that $z \preceq x \wedge y$. This completes the proof that $x \wedge y$ is the infimum in $(L, \preceq)$ of the set $\{x, y\}$.

Remark 2. *Given a lattice-based sum* $(L, \wedge, \vee) = \bigoplus_{\alpha\in\Lambda} (L_\alpha, \wedge_\alpha, \vee_\alpha)$. *The partial order relation* $\preceq$ *on the lattice* L *obtained by setting* $x \preceq y$ *in* L *if and only if* $x \wedge y = x$ *coincides with the partial order relation given by* (1). *One obtains the same partial order relation from the given lattice by setting* $x \preceq y$ *in* L *if and only if* $x \vee y = y$.

Proposition 7. *Let* $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ *be a semibounded* Λ*-sum family of posets with* (M) *condition, and let* $(L, \preceq) = \bigoplus_{\alpha\in\Lambda} (L_\alpha, \preceq_\alpha)$ *be its lattice-based sum. Then, for each* $x, y \in L$ *such that* $x \in L_\alpha$ *and* $y \in L_\beta$ *for some* $\alpha, \beta \in \Lambda$ *with* $\alpha \neq \beta$*, or such that* $x, y \in L_\alpha$ *for some* $\alpha \in \Lambda$ *with* x *or* y *are equal to one of the boundaries of* L_α*, the supremum and the infimum of the set* $\{x, y\}$ *exist in* $(L, \preceq)$.

Proof. Replace each L_δ by $L'_\delta = \{\perp_\delta, \top_\delta\}$. If L_δ is a singleton, then so is L'_δ, because $\perp_\delta = \top_\delta$. In case that $x \in L_\alpha$, replace L'_α by $L'_\alpha = L'_\alpha \cup \{x\}$. Hence each L'_α contains at most three elements and at least one element, that is either $L'_\alpha = \{\perp_\alpha = \top_\alpha\}$, $L'_\alpha = \{\perp_\alpha, \top_\alpha\}$, or $L'_\alpha = \{\perp_\alpha, z, \top_\alpha\}$ where $z \in \{x, y\}$. Trivially, each L'_α is a lattice and thus we are creating new Λ-sum family $\{(L'_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ but Λ-sum family of lattices for which $x, y \in L' = \bigcup_{\alpha\in\Lambda} L'_\alpha$ and the lattice-based sum construction in the sense of Definition 6 can be applied. Hence, by Theorem 6, the lattice-based sum $\bigoplus_{\alpha\in\Lambda} (L'_\alpha, \preceq_\alpha)$ is a lattice and hence the supremum and the infimum of the set $\{x, y\}$ exist.

4. Conclusions and future work

In this contribution, we submit to the reader a proposal of a new construction method for ordered structure which is promising for the further development of many-valued logics, generalizing both ordinal and horizontal sums. We generalized the well-known ordinal sum technique of posets to what we call lattice-based sum of posets by allowing for lattice ordered index set instead of linearly ordered index set, showing that the lattice-based sum of posets is again a poset. It is pointed out that our new approach extends also the horizontal sum based on an unstructured index set (i.e., any two distinct indices are incomparable). Moreover, we showed that if the summand posets are lattices, then the lattice-based sum will be a lattice provided that the summand lattices satisfy a condition (M), accordingly, any two distinct and incomparable elements which are not involved in the same summand lattice have both maximal and minimal indices. Perhaps, the most important consequence of such a sum-type technique on how to build new posets (lattices) from the fixed ones is that a new construction method arises.

Note that though a consecutive repetition of standard ordinal and horizontal sum constructions is covered by our approach, the opposite is not true. First of all, we can deal also with unbounded posets what is not the case of horizontal sums (common top and bottom elements of all involved posets are required by horizontal sum construction). Next, the consecutive repetition of mentioned classical construction has impact on the structure of the lattice-ordered index set, i.e., we have finitely many chains and unstructured parts, and techniques overcoming this defect will be superfluously costly (e.g., transfinite induction...).

These considerations would inevitably lead one into studying the expressive power of lattice-based sums. Thus, it might be worthwhile looking for a characterization of all lattice-based sums that can be obtained as ordinal and horizontal sums of the given summands. For this end, the relationship between series-parallel posets and N-free posets may prove useful, and articles [12, 13, 31] may be relevant. Also, it might be worthwhile looking for a characterization of posets that are not decomposable into a lattice-based sum in a nontrivial way. But first one should define precisely what is meant by a trivial or a nontrivial decomposition (see Remark 1).

One possible meaningful way (as suggested by an anonymous referee) to study the expressive power of lattice-based sums could be the following. We say that a class $\mathcal{C}$ of posets is *closed under taking lattice-based sums*, if for all $(\Lambda, \sqsubseteq) \in \mathcal{C}$ and for all $(L_\alpha, \preceq_\alpha) \in \mathcal{C}$ $(\alpha \in \Lambda)$ such that Λ is a lattice and $\{(L_\alpha, \preceq_\alpha)\}_{\alpha\in\Lambda}$ is a Λ-sum family, we have $\bigoplus_{\alpha\in\Lambda}(L_\alpha, \preceq_\alpha) \in \mathcal{C}$. Then we could ask the following questions. What are the classes of posets closed under taking lattice-based sums? Given a set $\mathcal{S}$ of posets, what is the smallest class of posets that contains $\mathcal{S}$ and is closed under taking lattice-based sums (i.e., the class generated by $\mathcal{S}$)? We suggest these questions – as well as the two mentioned in the previous paragraph – as a topic of future research.

Clearly, inspired by ideas of Clifford [6] (in the context of ordinal sums of abstract semigroups) and [15, 20, 21, 22, 29, 25, 28, 30] (ordinal sums of t-norms), further development of this approach could deal with the lattice-based sums of semigroups; then meet operations can be replaced by semigroup operations. This also allows us to study the theory of t-norms on bounded lattices from the point of view of lattice-based sums. We remark that other summand operations could also be taken into account (compare also, e.g., [10, 15, 20, 23]). These topics will be investigated in a future sequel to the present article.

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